

Operators with Periodic Hamiltonian Flows in Domains with the Boundary

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Abstract

We consider operators in the domains with the boundaries and derive sharp spectral asymptotics (containing non-Weyl correction) in the case when Hamiltonian flow is periodic.

Even if operator is scalar but not second order (or even second-order but there is an inner boundary with both refraction and reflection present) Hamiltonian flow is branching on the boundary and the notion of periodicity becomes more complicated.

0 Introduction

In section 6.2 of [Ivr2]¹⁾

we considered the case when there is no boundary and Hamiltonian flow is periodic and derive very sharp spectral asymptotics. Now we want to achieve similar results when there is a boundary. We do not expect asymptotics to be that sharp, but better than $O(h^{1-d})$ and have non-Weyl correction term. However the presence of the boundary, more precisely, branching of the rays on the boundary brings the new possibilities.

¹⁾ This article is a rather small part of the huge project to write a book and is just section 8.3 consisting entirely of newly researched results. section 6.2 roughly corresponds to Section 4.7 of its predecessor V. Ivrii [Ivr1]. External references by default are to [Ivr2].

Even if operator is scalar but not second order (or even second-order but there is an inner boundary with both refraction and reflection present) Hamiltonian flow is branching on the boundary and the notion of periodicity becomes more complicated.

For simplicity we consider only Schrödinger operators and derive sharp spectral asymptotics (containing non-Weyl correction) in the case when Hamiltonian flow is periodic.

1 Discussion and plan

We start our analysis in subsection 2 from the case when there is no branching and the Hamiltonian flow (with reflections) is periodic. Then if period is T_0 , $\Psi_{T_0} = I$ we have the same equality

$$(1.1) \quad e^{ih^{-1}T_0A}Q = e^{iT_0L}Q$$

as before where $\text{supp } q$ is disjoint from the boundary and only transversal to the boundary trajectories originate at $\text{supp } q$ and L is h -pseudo-differential operator. Then

$$(1.2) \quad F_{t \rightarrow h^{-1}\tau} \bar{\chi} \left(\frac{t}{T_0} - n \right) \Gamma(Q_{1x} u^t Q_{2y}) = (2\pi h)^{1-d} J(n) + O(h^{1-d+\delta})$$

with

$$(1.3) \quad J(n) = \int_{\Sigma_\tau} e^{in\ell} q_1 q_2 d\mu_\tau$$

as $\bar{\chi}$ is supported in $(-\frac{2}{3}, \frac{2}{3})$ and equals 1 in $(-\frac{1}{3}, \frac{1}{3})$ and $n \in \mathbb{Z}$, $|n| \leq h^{-\delta}$. Here as before $\Sigma_\tau = \{a(x, \xi) = \tau\}$ and $\mu_\tau = dx d\xi : da$ is the measure there, ℓ is the principal symbol of L .

If ℓ is “variable enough” on Σ_τ so the right-hand expression is decaying as $n \rightarrow \infty$ we can improve remainder estimate $O(h^{1-d})$ (to what degree depends on the rate of the decay). In section 6.2 essentially ℓ was defined by integral of the subprincipal symbol along periodic trajectories. Now however ℓ can pick up extra terms at the moment of reflection: for example, for Schrödinger operator increment of ℓ is 0 and π for Neumann and Dirichlet boundary condition respectively; however more general boundary condition brings increment which depends on the point $(x', \xi') \in T^*Y$ of reflection.

Then in section 3 we consider a more complicated case when two operators are intertwined through boundary condition and all Hamiltonian trajectories

of one of them are periodic but of another is not and, moreover, if the generic Hamiltonian billiard is periodic, it is in fact Hamiltonian billiard of the first operator. Then (1.1) fails but (1.2)–(1.3) is replaced by (1.2), (1.4)

$$(1.4) \quad J(n) = \int_{\Sigma_{1,\tau}} e^{in\ell - |n|\ell'} q_1 q_2 d\mu_{1,\tau}$$

where $\Sigma_{1,\tau}$ and $\mu_{1,\tau}$ correspond to the first operator and $\ell' \geq 0$ depends on the portion of energy which was whisked away along trajectory of the second operator at reflection point. Then if

$$(1.5) \quad \mu_{1,\tau}(\{(x, \xi) \in \Sigma_{1,\tau}, \ell' = 0, \nabla \ell = 0\}) = 0$$

we conclude that $J(n) = o(1)$ as $n \rightarrow \infty$ and we will be able to improve remainder estimate $O(h^{1-d})$ to $o(h^{1-d})$ and again there will be a non-Weyl correction term but it will depend also on ℓ and ℓ' .

Finally, in subsection 3 we consider even more complicated when two operators are intertwined through boundary condition and Hamiltonian trajectories of both of them are periodic and moreover after a while all trajectories issued from some point at T^*Y assemble there. Then we can construct a matrix symbol and its eigenvalues play a role of symbol ℓ .

2 Simple Hamiltonian flow

2.1 Inner asymptotics

Let \mathcal{U} be an open connected subset in T^*X disjoint from ∂X . Consider Hamiltonian billiards issued from \mathcal{U} , let Φ_t denote generalized Hamiltonian flow. Assume that

$$(2.1) \quad |\nabla_{x,\xi} a(x, \xi)| \geq \epsilon \quad \forall t \quad \forall (x, \xi) \in \Phi_t(\mathcal{U}),$$

$$(2.2) \quad |\phi(x)| + |\{a, \phi\}(x, \xi)| \geq \epsilon \quad \forall t \quad \forall (x, \xi) \in \Phi_t(\mathcal{U}),$$

$$(2.3) \quad \Phi_t(x, \xi) = (x, \xi) \quad \text{with } t = t(x, \xi) > 0 \quad \forall (x, \xi) \in \mathcal{U},$$

where $\phi \in \mathcal{C}^\infty(\bar{X})$, $\phi > 0$ in X and $\phi = 0$ on ∂X and

$$(2.4) \quad \text{Along } \Phi_t(\mathcal{U}) \text{ reflections are simple without branching, i.e. } \iota^{-1}\iota(y, \eta) \cap \{a(y, \eta) = \tau\} \text{ consists of two (disjoint) points } (y, \eta^\pm) \text{ such that } \pm\{a, \phi\} > 0 \text{ as } (y, \eta) \in \Phi_t(\mathcal{U}), y \in \partial X \text{ and } \tau = a(y, \eta).$$

Let us recall that $\iota : T^*X|_{\partial X} \rightarrow T^*\partial X$ is a natural map.

Then exactly as in the case without boundary

(2.5) One can select $t(x, \xi) = T(a(x, \xi))$ in \mathcal{U} with $T \in \mathcal{C}^\infty$ (so period depends on energy level only²⁾)

and

(2.6) As $(x, \xi) \in \mathcal{U}$ and $0 < t < T(x, \xi)$ $\Phi_t(x, \xi)$ meets ∂X exactly $(N - 1)$ times with $N = \text{const}$; so as $N \geq 1$ billiard consists of N segments with the ends on the boundary.

We can consider instead of A with domain

$$(2.7) \quad \mathfrak{D}(A) = \{u \in \mathcal{H}^m(X), \tilde{\partial} B u = 0\}$$

operator $f(A)$ with

$$(2.8) \quad f(\tau) = \int T^{-1}(\tau) d\tau$$

and introduce propagator $e^{ih^{-1}tf(A)}$ so that the Hamiltonian flow Ψ_t of $f(A)$ is periodic with period 1.

However now structure of $f(A)$ near ∂X is a bit murky; but it is not really serious obstacle since we can consider problem

$$(2.9) \quad (\check{f}(hD_t) - A)e^{ih^{-1}tf(A)} = 0,$$

$$(2.10) \quad \tilde{\partial} B e^{ih^{-1}tf(A)} = 0$$

where \check{f} is an inverse function. Therefore its Schwartz kernel $U(x, y, t)$ satisfies problem (2.9)–(2.10) with respect to x and transposed problem with respect to y .

Note that due to assumptions (2.1)–(2.2) $e^{ih^{-1}tf(A)}Q$ is a Fourier integral operator with the symplectomorphism Ψ_t provided Q is h -pseudo-differential operator with the symbol supported in \mathcal{U} . Further, due to (2.3), (2.8) $\Psi_1 = I$ and therefore $e^{ih^{-1}f(A)}Q$ is h -pseudo-differential operator. So, we arrive to

²⁾ But there could be subperiodic trajectories with periods $n^{-1}T(a(x, \xi))$, $n = 2, 3, \dots$. The structure of subperiodic trajectories described in subsection 6.2.5.

Proposition 2.1. *Let A be a Schrödinger operator with the scalar principal symbol $a(x, \xi)$ satisfying (2.1)–(2.3) and (2.8). Then*

$$(2.11) \quad e^{ih^{-1}f(A)-i\kappa} \equiv e^{ih^{-1}\varepsilon L} \quad \text{in } \mathcal{U}$$

with $\varepsilon = h$ and h -pseudo-differential operator L ; here κ is Maslov's constant and the principal symbol ℓ of L is defined by

$$(2.12) \quad e^{i\ell} = \prod_{1 \leq k \leq N} e^{i\ell_k} e^{i\ell'_k}$$

where ℓ_k corresponds to k -th segment (see theorem 6.2.3) and ℓ'_k corresponds to k -th reflection.

Example 2.2. Consider Schrödinger operator. Assume that at the reflection point $a(x, \xi) = \xi_1^2 + a(x', \xi')$, $b = \xi_1 + i\beta(\xi')$. Then

$$(2.13) \quad e^{i\ell'_k} = -((\tau - a')^{\frac{1}{2}} - i\beta)^{-1}((\tau - a')^{\frac{1}{2}} + i\beta)$$

with the right-hand expression calculated in the corresponding reflection point.

Taking $\beta = 0$ or $\beta = \infty$ (formally) we get $\ell'_k = 0$ and $\ell'_k = \pi$ for Neumann and Dirichlet boundary conditions respectively.

Then we can use all the local results of section 6.2, assuming that (2.11) holds with $h' \leq \varepsilon \leq h$ and derive local spectral asymptotics with the same precision (up to $O(h^{2-d})$) as there.

Corollary 2.3. *Let conditions of proposition 2.1 be fulfilled. Then asymptotics with the remainder $o(h^{1-d})$ holds provided*

$$(2.14) \quad \mu_\tau(\{(x, \xi) \in \Sigma_\tau, \nabla_{\Sigma_\tau} \ell(x, \xi) = 0\}) = 0$$

where ∇_{Σ_τ} means a gradient along Σ_τ ;

(ii) *Asymptotics with the remainder $O(h^{1-d+\delta})$ with small exponent $\delta > 0$ holds provided*

$$(2.15) \quad \mu_\tau(\{(x, \xi) \in \Sigma_\tau, |\nabla' \ell(x, \xi)| \leq \varepsilon\}) = o(\varepsilon^{\delta'})$$

with the small exponent $\delta' > 0$;

(iii) Furthermore asymptotics with the remainder $o(h^{1-d})$ holds as (2.1)–(2.2) are replaced by their non-uniform versions

$$(2.2)' \quad |\phi(x)| + |\{a, \phi\}(x, \xi)| > 0 \quad \forall t \quad \forall (x, \xi) \in \Phi_t(\mathcal{U} \cap \Sigma_\tau \setminus \Lambda_\tau),$$

where $\mu_\tau(\Lambda_\tau) = 0$.

We leave to the reader to formulate statement similar to (iii) but with the remainder estimate $O(h^{1-d+\delta})$.

Usually our purpose is the asymptotics with $Q_{1x} = \psi_1(x)$, $Q_{2y} = \psi_2(y)$ where ψ_j are smooth functions but not vanishing near the ∂X .

Corollary 2.4. *Asymptotics with the remainder estimate $o(h^{1-d})$ holds provided conditions (2.1), (2.2)', (2.3) and (2.14) are fulfilled in $\mathcal{U} = T^*X$, $Q_j = \psi_j \in \mathcal{C}^\infty(\bar{X})$ compactly supported.*

Proof. Due to corollary 2.3(iii) contribution of the zone $\{x_1 > \varepsilon\}$ to the remainder is $o_\varepsilon(h^{1-d})$ while contribution of zone $\{x_1 \leq \varepsilon\}$ to the remainder does not exceed $O(\varepsilon h^{1-d})$ as $\varepsilon \geq h$. \square

These arguments could be improved to $\varepsilon = h^\delta$ and the remainder estimate could be improved to $O(h^{1-d+\delta})$. We leave the precise statements and arguments to the reader. Our goal is to derive more sharp remainder estimate.

2.2 Asymptotics near boundary

Let us consider $X = \{x_1 > 0\}$ and \mathcal{U} open subset in $T^*\mathbb{R}_{x'}^{d-1} \times [0, \varepsilon]_{x_1} \times \mathbb{R}_\tau$ where $\varepsilon > 0$ is a very small constant. We are interested in

$$(2.16) \quad F_{t \rightarrow h^{-1}\tau} \bar{\chi}_T(t) \operatorname{Tr}(e^{ih^{-1}tf(A)} Q)$$

with $Q = Q(x, hD'_x, hD_t)$ with symbol supported in \mathcal{U} ; let us rewrite it as

$$F_{t \rightarrow h^{-1}\tau} \bar{\chi}_T(t) \operatorname{Tr}(e^{ih^{-1}(t-\bar{t})f(A)} Q' e^{ih^{-1}\bar{t}f(A)}) = F_{t \rightarrow h^{-1}\tau} \bar{\chi}_T(t) \operatorname{Tr}(e^{ih^{-1}tf(A)} Q')$$

with $Q' = e^{ih^{-1}\bar{t}f(A)} Q e^{-ih^{-1}\bar{t}f(A)}$.

Note that if (2.1), (2.2) are fulfilled, $\varepsilon > 0$ and $\bar{t} \geq C\varepsilon$ are small enough then $Q' = Q'(x, hD_x, hD_t)$ is h -pseudo-differential operator with the symbol supported in $\{\varepsilon'\bar{t} \leq x_1 \leq C'\bar{t}\}$ and we can rewrite

$$Q' = Q''(x, hD_x) + Q'''(x, hD_x, hD_t)(hD_t - f(A));$$

therefore we can rewrite (2.16) in the same form but with Q replaced by Q'' .

Now we can apply all the arguments of the previous subsection and of 6.2 and derive asymptotics with the remainder estimate as sharp as $O(h^{2-d})$ provided conditions (2.1)–(2.3) are fulfilled in the full measure as well as corresponding conditions of section 6.2 to ℓ . Here as usual we lift the points of $(y', \eta', \tau) \in T^*\partial X$ to $(y', \eta^\pm) \in \Sigma_\tau$.

So, we need to analyze only near tangent zone $\{x_1 \leq \varepsilon, |\{a, x_1\}| \leq \varepsilon\}$.

Unfortunately the theory of the manifolds with the boundaries and periodic Hamiltonian (or even geodesic) flow with reflections is completely undeveloped.

Let us assume that (2.1), (2.2)' and (2.3) are fulfilled for all $(x, \xi) \in T^*X : \tau_1 \leq a(x, \xi) \leq \tau_2$ with $\tau_1 < \tau_2$. Then trajectories tangent to ∂X cannot penetrate into X and therefore $x_1 = 0, \tau_1 \leq a'(x, \xi') \leq \tau_2$ is incompatible with $a'_{x_1}(x, \xi) < 0$.

Further, $x_1 = 0, \tau_1 \leq a'(x, \xi') \leq \tau_2$ is incompatible with $a'_{x_1}(x, \xi) > 0$ as well; really otherwise almost tangent to boundary billiards make very small jumps which contradicts to periodicity after N reflections with fixed N .

Therefore ∂X is *bicharacteristically flat*

$$(2.17) \quad x_1 = 0, \tau_1 \leq a'(x, \xi') \leq \tau_2 \implies a'_{x_1}(x, \xi) = 0.$$

Further, (2.3) implies that

(2.18) All solutions of equation

$$(2.19) \quad z'' + b(\Psi_t(x', \xi'))z = 0 \quad \text{with} \quad b(x', \xi) = \frac{1}{2}a'_{x_1 x_1}(0, x', \xi')$$

are T_0 -periodic for any $(x', \xi') \in T^*\partial X, \tau_1 \leq a'(x, \xi') \leq \tau_2$.

Therefore (2.3) implies that

$$(2.20) \quad \rho(x, \xi) \circ \Psi_t \asymp \rho \quad \text{with} \quad \rho = x_1 + |\{a, x_1\}| \quad \text{as} \quad \tau_1 \leq a(x, \xi) \leq \tau_2.$$

Consider zone $\mathcal{U}_\varepsilon = \{(x, \xi) : \rho(x, \xi) \asymp \varepsilon\}$. Blowing up $(x_1, \xi_1) \rightarrow \varepsilon^{-1}(x_1, \xi_1)$ we can prove easily that after this

$$(2.21) \quad \Psi_t(x, \xi) = (\Psi'_{t,\varepsilon}(x, \xi), \Psi''_{(x', \xi', t), \varepsilon}(x_1, \xi_1)),$$

with

$$(2.22) \quad \Psi'_{t,\varepsilon}(x, \xi) \sim \Psi'_{t,0}(x', \xi') + \sum_{n \geq 2} \varepsilon^n \Psi'_{t,(n)}(x, \xi),$$

$$(2.23) \quad \Psi''_{(x', \xi', t), \varepsilon}(x_1, \xi_1) \sim \sum_{n \geq 1} \varepsilon^n \Psi''_{(x', \xi', t), (n)}(x_1, \xi_1)$$

where $\Psi'_{t,0}$ is the Hamiltonian flow on $T^*\partial X$ and $\Psi''_{t,(1)}$ is the linearized with respect to (x_1, ξ_1) billiard flow (described by (2.19)). One can prove easily that $\Psi'_{t,(n)}$ and $\Psi''_{(x', \xi', t), (n)}$ are uniformly smooth.

Then in blown-up coordinates (2.11) holds provided $\varepsilon \geq h^{\frac{1}{2}-\delta}$ because $\hbar = \varepsilon^{-2}h$ and then we can apply all the previous arguments and derive asymptotics; the remainder estimate is as sharp as $O(h^{2-d})$ provided (in not blown-up coordinates)

$$(2.24) \quad |\nabla_{(x,\xi)} \ell(x, \xi)| \geq \epsilon_0 \quad \text{as} \quad x_1 \asymp \rho(x, \xi) \leq \epsilon_1;$$

$x_1 \asymp \rho(x, \xi)$ means exactly that $x_1 \geq \epsilon |\xi_1|$.

Therefore

Proposition 2.5. *Let A be a Schrödinger operator with the principal symbol $a(x, \xi)$ satisfying (2.1)–(2.3) and (2.17)–(2.18). Let the subprincipal symbol and the boundary condition be such that (2.24) is fulfilled. All these conditions are supposed to be fulfilled in $\{(x, \xi) : \rho(x, \xi) \leq \epsilon_0\}$.*

Then the contribution of zone $\{h^{\frac{1}{2}-\delta} \leq \rho(x, \xi) \leq \epsilon_2\}$ to the remainder is $O(h^{2-d})$.

Since the contribution of zone $\{\rho(x, \xi) \leq \varepsilon\}$ to the remainder does not exceed $C\varepsilon^2 h^{1-d}$ and we can take $\varepsilon = h^{\frac{1}{2}-\delta}$ we conclude that the contribution of zone $\{\rho(x, \xi) \leq \epsilon_2\}$ to the remainder is $O(h^{2-d-\delta})$. One can also estimate this way the contribution of subperiodic trajectories. Thus we arrive (details are left to the reader) to

Theorem 2.6. *Let A be a Schrödinger operator with the principal symbol $a(x, \xi)$ satisfying (2.1)–(2.3) and (2.17)–(2.18). Let the subprincipal symbol and the boundary condition be such that (2.24) is fulfilled. All these conditions are supposed to be fulfilled in $\{(x, \xi) : \rho(x, \xi) \leq \epsilon_0\}$.*

Then contribution of zone $\{\rho(x, \xi) \leq \epsilon_2\}$ to the remainder is $O(h^{2-d-\delta})$ while its contribution to the principal part of the asymptotics is given by the standard two term Weyl formula plus a correction term constructed according to section 5.2 for symbol ℓ .

Problem 2.7. Recover $O(h^{2-d})$ remainder estimate. This should not be extremely hard especially if there are no subperiodic trajectories but definitely worth of publishing.

Example 2.8. Consider Laplacian $h^2\Delta$ on the standard hemisphere $\mathbb{S}_+^d := \{|x| = 1, x_1 > 0\}$ in \mathbb{R}^{d+1} or harmonic oscillator $h^2\Delta + |x|^2$ on the standard half-space $\mathbb{R}_+^d = \{x_1 > 0\}$.

(i) Consider boundary operator of example 2.2. Then $N = 2$ and $\ell(z, \tau) = \ell'(z, \tau) + \ell'(\varrho(z), \tau)$ where $z = (x', \xi')$, $\varrho(z)$ is the *antipodal point* and ℓ' is defined by (2.13). Then one can express conditions to ℓ as conditions to β ; we leave exact statements to the reader;

(ii) Consider Dirichlet or Neumann boundary conditions. Then as perturbation is hx_1 one can calculate easily that $\ell = \kappa\rho + O(\rho^2)$ where ρ is the incidence angle of trajectory and $\kappa > 0$. Then condition (2.24) is fulfilled.

2.3 Discussion

As we mentioned almost nothing is known about manifolds with all billiards closed; we are discussing geodesic billiards but one can ask the same questions about Hamiltonian billiards or Hamiltonian billiards associated with the Schrödinger operator; in the two last cases phase space S^*X is replaced by the portion of the phase space $\{\tau_1 \leq a(x, \xi) \leq \tau_2\}$. We assume that the phase space is connected:

Problem 2.9. (i) So far our only examples are a standard hemisphere \mathbb{S}_+^d and a standard half-space \mathbb{R}_+^d (see figure below); are any other there really different examples?

(ii) In particular, are there manifolds with $N \neq 2$ (i.e. with $N = 1, 3, 4, \dots$)? We can consider N copies X_1, \dots, X_N of the standard quarter-sphere $\mathbb{S}^2 \cap \{0 \leq \theta \leq \frac{\pi}{2}\} \cap \{0 \leq \phi \leq \pi\}$, then glue $\{X_k, \phi = \pi\}$ with $\{X_{k+1}, \phi = 0\}$ for $k = 1, \dots, N$, where $X_{N+1} := X_N$. Then for resulting manifold we have N reflections but for $N \neq 2$ there will be singularity at North Pole.

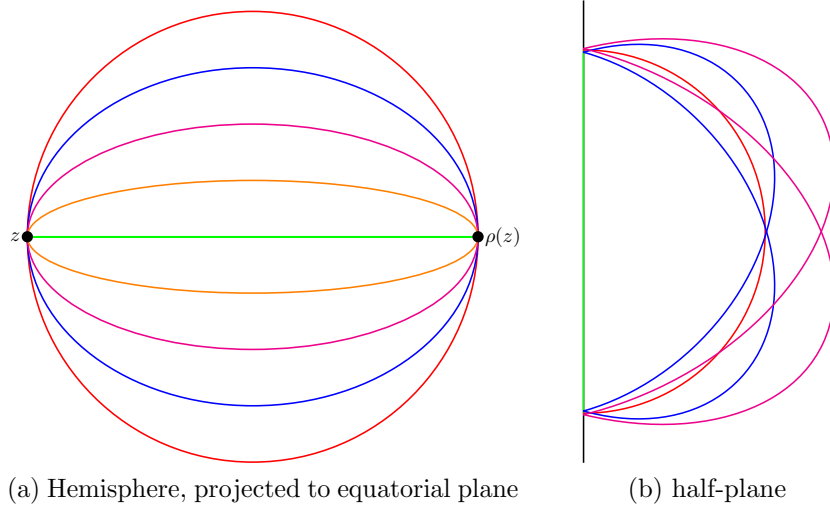


Figure 1: Different billiards (actually, their x -projections) are shown by different colors, they reflect from the boundary at two mutually antipodal points.

- (iii) In particular, are there manifolds with ∂X which is not connected?
- (iv) In particular, are there manifolds with subperiodic billiards and (or) with subperiodic boundary trajectories?

3 Branching Hamiltonian flow with “scattering”

Let us consider two manifolds X_1 and X_2 glued along the connected component Y of the boundary. We consider Schrödinger operators A_j on X_j and these operators are intertwined through the boundary conditions. We assume that Hamiltonian flows have simple (no branching) reflections on each of them, so branching comes from “reflection-refraction”. Further we are interested in the case when Hamiltonian flow on X_1 is periodic while on X_2 majority of trajectories are not periodic.

3.1 Analysis in X_2

Since we do not know any other examples of manifolds with periodic Hamiltonian flows with reflections but hemisphere or half-space and $N = 2$ then we assume that

(3.1) For $z \in T^*Y$ $\Phi_{1,\tau}(z) = \varrho(z)$ ³⁾, ϱ is an antipodal map on T^*Y .

We also assume that $\Phi_{1,\tau}$ and $\Phi_{2,\tau}$ commute

$$(3.2) \quad \Phi_{1,\tau} \circ \Phi_{2,\tau} = \Phi_{2,\tau} \circ \Phi_{1,\tau}.$$

Then for points of $\Sigma_{2,\tau}$ periodicity properties of the branching Hamiltonian flow with reflections Ψ_t coincide with those of $\Psi_{2,t}$.

Now we need to impose condition to the boundary operator except that $\{A, B\}$ is self-adjoint. Namely, consider a point $z = (x', \xi') \in T^*Y$, and consider an auxiliary problem

$$(3.3) \quad a_j(x', D_1, \xi')u_j = 0 \quad j = 1, 2.$$

$$(3.4) \quad \partial b(x', D_1, \xi')u = 0$$

(where $u = (u_1, u_2)$) and consider solutions which are combinations of exponents $e^{i\lambda_j x_1}$ where for each j either $\text{Im } \lambda_j > 0$ or $\text{Im } \lambda_j < 0$ and $\{a_j, x_1\}(x', \lambda_j, \xi') > 0$. Denote by Λ the set of points for which such non-trivial solutions exist. We assume that

$$(3.5) \quad \text{mes}_{T^*Y} \Lambda = 0.$$

Then we arrive to the following

Proposition 3.1. *Let in the described setup conditions (3.1), (3.2) and (3.5) be fulfilled.*

*Let Q_1, Q_2 be h -pseudo-differential operators with the symbols supported in T^*X_2 and*

$$(3.6) \quad \mu_{2,\tau}(\Pi_{2,\tau} \cap \text{supp } q_1 \cap \text{supp } q_2) = 0$$

where $\Pi_{2,\tau}$ is the set of point $(y, \eta) \in \Sigma_{2,\tau}$ periodic with respect to $\Psi_{2,t}$.

Then for $\Gamma(Q_{1x}e(\cdot, \cdot, \tau)^t Q_{1y})$ the standard Weyl two-term asymptotics holds with the remainder estimate $o(h^{1-d})$.

³⁾ We define $\Phi_{j,\tau}$ in the following way: we lift $z \in T^*Y$ to $z \in T_j^X|_Y \cap \Sigma_{j,\tau}$ with $\{a_j, x_1\}(z) > 0$, launch trajectory forward until it hits the boundary at $(y', \eta) \in T^*X_j|_Y$ and then project to T^*Y .

Remark 3.2. (i) Note that

$$(3.7) \quad \mu_{j,\tau}(\Lambda_{j,\tau}) = 0$$

where $\Lambda_{j,\tau} = \Lambda'_{j,\tau} \cup \Lambda''_{j,\tau}$, $\Lambda'_{j,\tau}$ is the set of dead-end points of billiard (not generalized billiard) flow and $\Lambda''_{j,\tau}$ is the set of points $(y, \eta) \in \Sigma_{j,\tau}$ such that $\iota_j \Psi_{j,t}(y, \eta) \in \Xi_{3-j,\tau}$ where $\Xi_{k,\tau} = \iota_k \{(x, \xi) \in \Sigma_{k,\tau}, \{a_k, x_1\} = 0\}$.

(ii) Note that the grows properties of the branching Hamiltonian flow with reflections Ψ_t coincide with those of $\Psi_{2,t}$ as long as we ensure that along billiards we do not approach to glancing rays on X_1 :

$$(3.8) \quad \begin{aligned} \{a_1, x_1\}(x, \xi) &\geq h^{\delta_1} \quad \forall (x, \xi) \in \iota_1^{-1} \iota_2 \Psi_{2,t}(y, \eta) \cap \Sigma_{1,\tau} \\ \forall (y, \eta) \in \text{supp } q_2 \quad \forall t : \pm t \in [0, T] \quad \Psi_{2,t}(y, \eta) &\in T^*X_2|_Y \end{aligned}$$

where $T = T(h)$. Then in assumptions of section 7.4 one can recover similar results: namely that two-term Weyl asymptotics holds with the remainder estimate $O(T(h)^{-1}h^{1-d})$ which is usually $O(h^{1-d}|\log h|^{-1})$ or $O(h^{1-d+\delta})$ (depending on the growth and non-periodicity conditions to $\Psi_{2,t}$. Exact statement and proof are left to the reader (which is a relatively easy problem).

(iii) In (ii) we need also to assume a more sharp version of (3.5): namely if we consider approximate solutions to (3.3)–(3.4) with precision ε we get set $\Lambda_{\tau,\varepsilon}$ and we need to impose condition $\text{mes } \Lambda_{\tau,\varepsilon} = o(\varepsilon^\delta)$ or $\text{mes } \Lambda_{\tau,\varepsilon} = o(|\log \varepsilon|^{-1})$ as $\varepsilon \rightarrow +0$.

(iv) In subsubsection “7.4.3” of subsection 7.4.3 we did not consider points $z \in T^*Y$ where some of operators a_j are elliptic on $\iota^{-1}z$ rather than hyperbolic. If we want to consider such points we need to assume (3.5) or more sharp version of it described in (iii).

(v) (3.8) is guaranteed if either a_2 is cylindrically symmetric on T^*X_2 or if there no internal reflection for billiards from X_2 at all:

$$(3.9) \quad \begin{aligned} |\{a_1, x_1\}(x, \xi)| &\geq \epsilon |\{a_2, x_1\}(x, \eta)|^l \\ \forall (x, \xi) \in T^*X_1|_Y \cap \Sigma_{1,\tau} \quad \forall (x, \eta) \in T^*X_2|_Y \cap \Sigma_{2,\tau} : \iota_1(x, \xi) &= \iota_2(x, \eta). \end{aligned}$$

3.2 Analysis in X_1

However if operators Q_1 and Q_2 have symbols supported in X_1 situation is very different: trajectories of a_1 are periodic but the typical closed branching billiard is the one which does not contain segments of Hamiltonian trajectories of a_2 . More precisely, due to conditions

$$(3.10) \quad \mu_{2,\tau}(\Pi_{2,\tau}) = 0$$

and (3.7) we conclude that

$$(3.11) \quad \mu_{1,\tau}(\Pi'_{1,\tau}) = 0$$

where $\Pi'_{1,\tau}$ is the set of points $z \in \Sigma_{1,\tau}$ such that there exists a billiard trajectory of Ψ_t starting and ending in z and which is not entirely billiard of $\Psi_{1,t}$.

Then for arbitrarily large $T > 0$ and arbitrarily small $\varepsilon > 0$ there exists a set $\mathcal{U} = \mathcal{U}_{T,\varepsilon} \subset T^*X$ such that

$$(3.12) \quad \mu_{1,\tau}(\Sigma_{1,\tau} \setminus \mathcal{U}) \leq \varepsilon,$$

$$(3.13) \quad \text{dist}(\mathcal{U}, \Pi'_{1,\tau,T} \cup \Lambda_{1,\tau,T}) \geq \gamma$$

$$(3.14) \quad \text{dist}(Y, \pi_x \mathcal{U}) \geq \gamma$$

with $\gamma = \gamma(T, \varepsilon) > 0$ where $\Pi'_{1,\tau,T}$ is the set of points $z \in \Sigma_{1,\tau}$ such that there exists a billiard trajectory of Ψ_t of the length not exceeding T , starting and ending in z and which is not entirely billiard of $\Psi_{1,t}$ and $\Lambda_{1,\tau,T}$ is the set of points $z \in \Sigma_{1,\tau}$ such that billiard trajectory of a_1 hits the boundary at point of $\Lambda_{2,T,\tau}$. As usual, these sets increase as T increases, their unions (with respect to T) coincide with $\Pi'_{1,\tau}$ and $\Lambda_{1,\tau}$ respectively and $\Lambda_{1,\tau,T}$ and $\Lambda_{1,\tau,T} \cap \Pi'_{1,\tau,T}$ are closed sets.

Therefore there exists operator $Q = Q_{T,\varepsilon}$ such that $\text{supp } q \subset \mathcal{U}$ and

$$(3.15) \quad |\Gamma(Q_{1x}e(.,.,\tau)^t((I-Q)Q_2)_y) - \kappa'_0 h^{-d} - \kappa'_1 h^{1-d}| \leq C_0 \varepsilon h^{1-d}$$

with $\kappa'_j = \kappa'_{j,Q_1,Q_2,Q}(\tau)$.

This estimate holds for any fixed τ satisfying (3.10). Therefore we need to consider $\Gamma(Q_{1x}e(.,.,\tau)^t Q_{2y})$ with $\text{supp } q_2 \subset \mathcal{U}$.

We can assume without any loss of the generality that (3.13), (3.14) are fulfilled for all τ . Then for $\varepsilon \leq |t| \leq T$

$$(3.16) \quad \Gamma(Q_{1x}U(.,.,t)^t Q_{2y}) \equiv \Gamma(Q_{1x}Z(.,.,t)^t Q_{2y})$$

where as $\pm t > 0$ $Z(t)$ is Schwartz kernel of “approximate propagation with scattering semigroup” $Z(t)$ constructed in the following way (for a sake of simplicity in notations we consider $t > 0$):

- (i) We represent $Q_2 = Q_{2,1} + \cdots + Q_{2,N}$ where symbols $Q_{2,\nu}$ have small supports;
- (ii) For each ν we select $0 = t_{0,\nu} < \cdots < t_{M,\nu} = t$ in such a way that for any $z \in \text{supp } q_{2,\nu}$ $\Psi_{t'}(z)$ is disjoint from Y as $t' = t_{j,\nu}$ and $\Psi_{t'}(z)$ hits Y no more than once as $t_{j,\nu} \leq t' \leq t_{j+1,\nu}$;
- (iii) We set

$$Z(t)Q_{2,\nu} = \left(\prod_{0 \leq j \leq M-1} \psi e^{ih^{-1}(t_{j+1,\nu} - t_{j,\nu})f(A)} \right) \cdot Q_{2,\nu}$$

where $\psi \in \mathcal{C}^\infty$ is supported in $\{x \in X_1, \text{dist}(x, Y) \geq \gamma'\}$ and equals 1 in $\{x \in X_1, \text{dist}(x, Y) \geq 2\gamma'\}$.

Then

$$(3.17) \quad Z(t_1 + t_2)Q_2 \equiv Z(t_2)Z(t_1)Q_2 \quad \text{as } \pm t_1 > 0, \pm t_2 > 0$$

and

$$(3.18) \quad Z(t)^* \equiv Z(-t)$$

and due to periodicity of $\Psi_{1,t}$ with period T_0 $Z(\pm T_0)$ are h -pseudo-differential operators.

Remark 3.3. In contrast to what we had before they are not necessarily unitary because *reflection coefficients* are not unitary anymore. More precisely,

- (i) If $\Psi_{1,t}$ hits $T^*X|_Y$ at points (x', ξ) with $(x', \xi') \notin \iota_2 \Sigma_{2,\tau}$ reflection coefficient $\varkappa_{11}(x', \xi', \tau)$ still satisfies $|\varkappa_{11}(x', \xi', \tau)| = 1$.
- (ii) However if $\Psi_{1,t}$ hits $T^*X|_Y$ at points (x', ξ) with $(x', \xi') \in \iota_2 \Sigma_{2,\tau}$ then reflection-refraction matrix $\varkappa(x', \xi', \tau) = (\varkappa_{jk}(x', \xi', \tau))_{j,k=1,2}$ is unitary but reflection coefficient $\varkappa_{11}(x', \xi', \tau)$ satisfies only $|\varkappa_{11}(x', \xi', \tau)| \leq 1$ and the equality means exactly that $\varkappa_{12}(x', \xi', \tau) = \varkappa_{21}(x', \xi', \tau) = 0$ (then $|\varkappa_{22}(x', \xi', \tau)| = 1$ as well).

So the principal symbol of $Z(T_0)$ is the product of $e^{i\ell_0(x, \xi)}$ where $\ell_0(x, \xi)$ is given by the usual formula for manifolds without boundary (but with

trajectories replaced by billiards) and also of reflection coefficients coefficients in the point of reflection. If the reflection coefficients do not vanish we can rewrite the answer in the form

$$(3.19) \quad \mathbf{Z}(T_0) \equiv e^{i\varepsilon h^{-1}L}, \quad \mathbf{Z}(-\varepsilon h^{-1}T_0) \equiv e^{-iL^*}$$

with h -pseudo-differential operator L , $L + L^* \leq 0$ and $\varepsilon = h$ (but in calculations below we are not assuming this).

We will use this formula even if the coefficients vanish: in this case we simply formally allow $\text{Im } \ell = +\infty$.

Then as $\bar{\chi} \in \mathcal{C}_0^\infty[\varepsilon - 1, 1 - \varepsilon]$

$$(3.20) \quad F_{t \rightarrow h^{-1}\tau} \bar{\chi}_{T_0}(t - nT_0) \Gamma(Q_{1x} Z(\cdot, \cdot, t) {}^t Q_{2y}) = \\ h^{1-d} J(h, \varepsilon, n, \tau) \cdot \widehat{\bar{\chi}}\left(\frac{\tau}{h}\right) + O(h^{2-d}),$$

with

$$(3.21) \quad J(h, \varepsilon, n, \tau) = (2\pi)^{1-d} \int_{\Sigma_{1,\tau}} e^{ih^{-1}((\varepsilon \text{Re } \ell - \tau)n + \varepsilon \text{Im } \ell |n|)} d\mu_{1,\tau}.$$

We can see easily that

$$(3.22) \quad J(h, \varepsilon, n, \tau) = o(1) \text{ as } n \rightarrow \infty \text{ provided}$$

$$(3.23) \quad \mu_\tau(\{(x, \xi) \in \Sigma_\tau : \text{Im } \ell(x, \xi) = \nabla \text{Re } \ell(x, \xi) = 0\}) = 0.$$

On the other hand, note that

$$(3.24) \quad \sum_{n \in \mathbb{Z} \setminus 0} e^{ih^{-1}(n\alpha - |n|\beta)} = \frac{2e^{-\beta} \cos \alpha - 2e^{-2\beta}}{1 - 2e^{-\beta} \cos \alpha + e^{-2\beta}} \quad \text{as } \beta > 0$$

and formally we can extend this for $\beta = 0$ as well.

Then after summation with respect to n for $\beta = 0$ we get exactly $\partial_\alpha \Upsilon(\alpha)$ with Υ defined by (6.2.63) and for $\beta > 0$ we get that the expression (3.24) is equal to $\partial_\beta \Upsilon *_\alpha (2\pi)^{-1} G(\alpha, \beta)$ where $G(\alpha, \beta) = 2\beta(\alpha^2 + \beta^2)^{-1}$.

Note that

(3.25) Function

$$(3.26) \quad \Upsilon(\alpha, \beta) := \Upsilon *_{\alpha} G(\alpha, \beta)$$

is harmonic function as $\beta > 0$, coincides with $\Upsilon(\alpha)$ as $\beta = 0$, is $o(1)$ as $\beta \rightarrow +\infty$ and is periodic with respect to α .

Taking in account results of this subsection we arrive to

Theorem 3.4. *Let in the described setup conditions (3.1), (3.2), (3.5), (3.10) and (3.23) be fulfilled.*

Then asymptotics

$$(3.27) \quad \Gamma e(., ., \tau) = \kappa_0(\tau) h^{-d} + \left(\kappa_1(\tau) + \int_{\Sigma_{\tau}} \Upsilon(h^{-1}(\varepsilon \operatorname{Re} \ell - \tau), -h^{-1} \varepsilon \operatorname{Im} \ell) d\mu_{\tau} \right) + o(h^{1-d})$$

holds.

Problem 3.5. Calculate all coefficients \varkappa_{jk} for two Schrödinger operators A_1 and A_2 .

We will solve this problem in the very special case:

Example 3.6. Assume that (at the given point $z \in T^*Y$) $a_j = c_j^2 |\xi|^2$ and the boundary condition is

$$(3.28) \quad u_2 = \alpha u_1,$$

$$(3.29) \quad D_1 u_2 = -\beta D_1 u_1$$

(recall that $x_1 > 0$ in both X_1, X_2 ; in more standard notations of $x_1 < 0$ in X_2 one needs to skip “−”). Then $\{A, B\}$ to be self-adjoint requires $\alpha\beta^\dagger = c_1^2 c_2^{-2}$. We do not consider gliding points since their measure is 0.

(i) Consider point ξ' with $c_1^2 |\xi'|^2 < \tau < c_2^2 |\xi'|^2$. One can prove easily that

$$(3.30) \quad \varkappa_{11}(\xi', \tau) = e^{2i\varphi}$$

with

$$(3.31) \quad \varphi(\xi', \tau) = \arctan \left(|\beta|^{-2} c_2^{-2} c_1^2 (c_2^2 |\xi'|^2 - \tau)^{\frac{1}{2}} (\tau - c_1^2 |\xi'|)^{-\frac{1}{2}} \right)$$

and therefore assumption (3.23) is fulfilled provided $c_1 \neq c_2$, c_j, α, β are symmetric (i.e. $c_j \varrho = c_j$) and the subprincipal symbol is 0. If $c_1 > c_2$ this case does not appear.

(ii) Consider point ξ' with $c_j^2 |\xi'|^2 < \tau$ for both $j = 1, 2$. One can prove easily that

$$(3.32) \quad \varkappa_{11}(\xi', \tau) = (\omega - 1)(\omega + 1)^{-1}$$

with

$$(3.33) \quad \omega(\xi', \tau) = |\beta|^{-2} c_2^{-2} c_1^2 (\tau - c_2^2 |\xi'|^2)^{\frac{1}{2}} (\tau - c_1^2 |\xi'|^2)^{-\frac{1}{2}}$$

and $|\varkappa_{11}| < 1$ so assumption (3.23) is fulfilled.

The following problems seem to be rather straightforward but not easy and worth publishing:

Problem 3.7. Under stronger conditions to $\Psi_{2,t}$ and ℓ (see (2.15)) prove remainder estimate $O(h^{1-d+\delta})$.

Problem 3.8. Prove the same results as Hamiltonian billiards of \mathbf{a}_1 are assumed to be periodic only on one energy level τ rather than on neighboring levels.

4 Two periodic flows

Now we consider the case of both flows Ψ_{1t} and Ψ_{2t} generated by $f(\mathbf{a}_1)$ and $f(\mathbf{a}_2)$ being periodic and satisfying (3.1) as in the following figures:

4.1 Examples and discussion

Let us start from an example:

Example 4.1. Consider two Schrödinger operators in $\mathbb{R}_+^d = \mathbb{R}^+ \times \mathbb{R}^{d-1}$ with the principal symbols

$$(4.1) \quad a_j(x, \xi) = \frac{1}{2} \omega_j^2 (|\xi|^2 + |x|^2 - E_j)$$

intertwined through boundary conditions. Then $T_j = 2\pi\omega_j^{-2}$.

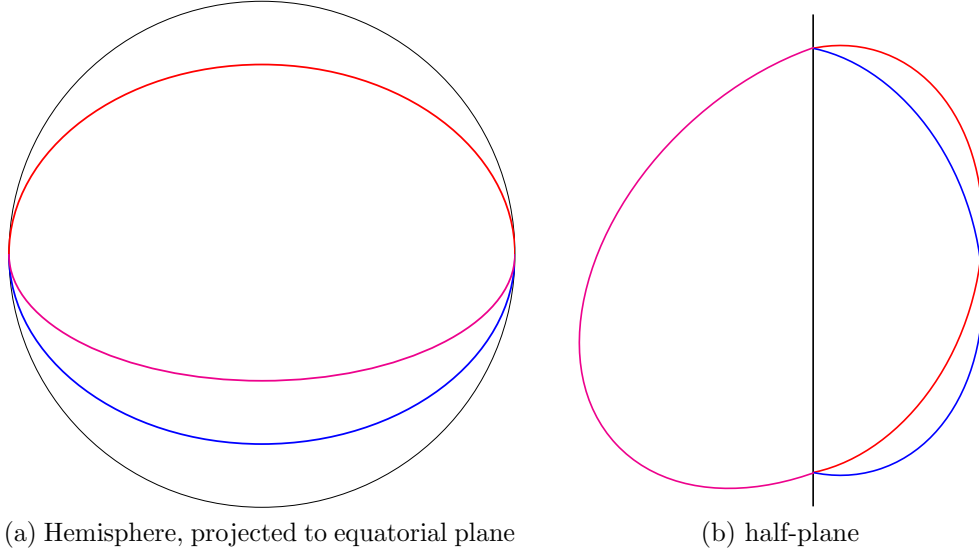


Figure 2: x -projection of billiard trajectory on two half-planes, glued together: original (red), reflected (blue), refracted (magenta)

We are interested in the number $N_h^-(0)$ of the negative eigenvalues of operator A_B . In order to consider the eigenvalue counting function $N_h^-(\lambda)$ for arbitrary spectral parameter λ one needs to redefine

$$(4.2) \quad E_j := E_j + 2\omega_j^2 \lambda.$$

But $N^-(0)$ is also the maximal dimension of the *negative subspace* of the operator A which does not change if we replace A by $J^{-\frac{1}{2}}AJ^{-\frac{1}{2}}$ with the positive self-adjoint operator J and the latter equals to $N_h^-(0)$ for the latter operator. Picking up $J = J(x)$ equal ω_j^2 in X_j we arrive to the case of $\omega_j = 1$ and $T_j = 2\pi$. Therefore in this example one can make periods equal $T_1 = T_2$. The dependence of ω_j will come through E_j redefined by (4.2).

There are however many deficiencies of the described approach; most important is that it does not work with the following example (unless some unnatural conditions to ω_j , E_j are imposed):

Example 4.2. Consider two operators on hemisphere \mathbb{S}_+^d with the principal symbols

$$(4.3) \quad a_j(x, \xi) = \omega_j^2 (|\xi|^2 - E_j)$$

where $|\xi|$ is calculated in the standard Riemannian metrics.

The main obstacle here is that multiplying operator by $J^{-\frac{1}{2}}$ from both sides and calculating $f(A)$ do not play well together.

4.2 Reduction to the boundary

First of all notice that the contribution to the remainder of the domain $\{z, \text{dist}(z, T^*Y) \leq \varepsilon_0\}$ does not exceed $C_0 \varepsilon_0^2 h^{1-d}$ and therefore one needs to consider the contribution of $\{z, \text{dist}(z, T^*Y) \geq \varepsilon_0\}$ with arbitrarily small but fixed ε_0 .

In this zone one can consider $U(x, y, t)$ as the solution of the Cauchy problem with respect to x_1 (or y_1) with data at $\{x_1 = 0\}$ (or $\{y_1 = 0\}$). Rewriting therefore $U = R_x U^t R_y$ where R is an operator resolving this Cauchy problem is given by an oscillatory integral we can rewrite

$$(4.4) \quad \Gamma(U(., ., t) {}^t Q_y) \equiv \Gamma'(\mathcal{Q}(x', hD'_x, hD_t, h)V)$$

with h -pseudo-differential operator R and $V = \check{\partial}_{x, m-1} \check{\partial}_{y, m-1} U$ the Cauchy data for U (assuming that m is an order of A).

One can see easily that as Q is an operator with the symbol supported in T^*X_j , the principal symbol of \mathcal{Q} at (x', ξ, τ) is defined as an averaging of Q along $\Psi_{jt}(z_j)$ with $t \in [0, t_j(z_j)]$ where $z_j = \iota^{-1}(x', \xi') \cap \Sigma_{j\tau}$ and $t_j(z_j)$ is the time of the next hit of the boundary.

Then instead of continuous family of Fourier integral operators $e^{ih^{-1}tA}$ we can consider a discrete family \mathcal{F}^n defined by the following way: consider $v_j = \check{\partial}_{m-1} u_j = (v_j^-, v_j^+)$ with v_j^\mp corresponding to incoming and outgoing solutions.

We have two types of trajectories: those in $\Sigma_{j\tau}$ which hit $T^*X|_Y$ at points elliptic for $(a_{3-j} - \tau)$ where $j = 1, 2$ and those which hit $T^*X|_Y$ at points hyperbolic for $(a_k - \tau)$ for both $k = 1, 2$. The analysis of the former is of no different from what we have seen in the previous subsection: we just consider Ψ_{jt} and construct the (real-valued) symbol ℓ as long as we assume that $(x', \xi) \notin \Lambda$ where Λ is defined in the paragraph preceding (3.5).

The analysis of the latter is more interesting. Therefore as before we have a matrix of reflection-refraction $(\varkappa_{jk})_{j,k=1,2}$ which is a symbol defined on $\mathcal{U} \times (\tau - \varepsilon', \tau + \varepsilon')$ where \mathcal{U} is a zone described above. This matrix is

unitary in the norm

$$(4.5) \quad \|V\| = \left(\sum_j (|V_j^+|^2 + |V_j^-|^2) |\{a_j, x_1\}|_{x_1=0} \right)^{\frac{1}{2}}.$$

We also have diagonal unitary matrices $M'_\nu = \text{diag}(e^{i\ell_{1\nu}}, e^{i\ell_{2\nu}})$ with $\ell_{j\nu}$ calculated along ν -th leg of the closed trajectory of Ψ_{jt} with $t = \frac{1}{2}T_0$ and $\nu = 1, 2$.

Consider closed branching billiard originated from point $z \in T^*Y$; first there are two trajectories (in X_1 and X_2) both hitting $T^*X|_Y$ at $\varrho(z)$ and then there are two trajectories (in X_1 and X_2) returning to z . Let us recall that $\varrho(z)$ is an antipodal point.

Therefore branching billiard is characterized by an unitary (in (4.5)-norm) matrix

$$(4.6) \quad M(z, \tau) = \varkappa(z, \tau) M'(z, \tau) \varkappa(\varrho(z), \tau) M'(\varrho(z), \tau).$$

4.3 Analysis of the evolution

One can prove easily that modulo $O_T(h^{2-d})$

$$(4.7) \quad F_{t \rightarrow h^{-1}\tau} \chi_T(t) \Gamma'(\mathcal{Q}(x', hD'_x, hD_t, h)V) \equiv \\ (2\pi h)^{1-d} \chi_T(T_1 + \dots + T_n) \iint \sum_{N \neq 1} \sum_{j=(j_1, \dots, j_N) \in \{1, 2\}^N} \\ M_{j_1 j_2} M_{j_2 j_3} \dots M_{j_{N-1} j_N} e^{-ih^{-1}\tau(T_{j_1} + \dots + T_{j_N})} \mathcal{Q}_{j_N j_1} dx' d\xi'$$

where $\chi \in \mathcal{C}_0^\infty$ is supported in $[\frac{1}{2}, 1]$ and symbols M_{jk} and \mathcal{Q} and T_j are calculated at (x', ξ', τ) where $T_j = T_j(\tau)$ since all trajectories are periodic on energy levels close to τ .

One can rewrite terms with equal N as

$$(4.8) \quad (2\pi h)^{1-d} \int \hat{\chi}(\lambda) e^{iT^{-1}\lambda(T_{j_1} + \dots + T_{j_N})} \iint \sum_{j=(j_1, \dots, j_N) \in \{1, 2\}^N} \\ M_{j_1 j_2} M_{j_2 j_3} \dots M_{j_{N-1} j_N} e^{-ih^{-1}\tau(T_{j_1} + \dots + T_{j_N})} \mathcal{Q}_{j_N j_1} dx' d\xi' d\lambda$$

which in turn equals

$$(4.9) \quad (2\pi h)^{1-d} \int \hat{\chi}(\lambda) \iint \text{tr}(S(\lambda, \tau, x', \xi')^N \mathcal{Q}) dx' d\xi' d\lambda$$

where

$$(4.10) \quad S(\lambda T^{-1}, \tau, x', \xi') = \begin{pmatrix} e^{i(\lambda T^{-1} + \tau h^{-1})T_1} m_{11} & e^{i(\lambda T^{-1} + \tau h^{-1})T_1} m_{12} \\ e^{i(\lambda T^{-1} + \tau h^{-1})T_2} m_{21} & e^{i(\lambda T^{-1} + \tau h^{-1})T_2} m_{22} \end{pmatrix}$$

and $M = (m_{jk})$.

Consider eigenvalues of this matrix; as $T \rightarrow \infty$ they tend to eigenvalues of the same matrix with $\mu = 0$. If

$$(4.11) \quad \text{mes}\{(x', \xi') : \rho \in \text{Spec}(S(0, \tau, x', \xi'))\} = o(1) \\ \forall \tau \in [\bar{\tau} - \epsilon h T^{-1}, \bar{\tau} + \epsilon h T^{-1}] \quad \text{as } T \rightarrow \infty$$

then this term is $o(h^{1-d})$ and therefore we estimated expression (4.7) by $o(Th^{1-d})$ which in the end of the day returns remainder estimate $o(h^{1-d})$ with the Tauberian main part.

Note that instead of S we can consider matrix

$$(4.12) \quad \tilde{S}(\tau, x', \xi') = \begin{pmatrix} e^{i\tau h^{-1}T^*} m_{11} & m_{12} \\ m_{21} & e^{-i\tau h^{-1}T^*} g_{22} \end{pmatrix}$$

with $T^* = \frac{1}{2}(T_1 - T_2)$ which is different from $S(0, \cdot, \cdot, \cdot)$ by factor $e^{\frac{1}{2}ih^{-1}\tau(T_1 + T_2)}$.

Consider correction to the main part; we are interested at $\tau = 0$; so we integrate (4.7) with $\chi = 1$ from $\tau = -\infty$ to $\tau = 0$ resulting in (after multiplication by h^{-1} and modulo $o(h^{1-d})$)

$$(4.13) \quad (2\pi h)^{1-d} \chi_T(T_1 + \dots + T_n) \iint \sum_{N \neq 1} \sum_{j \in \{1,2\}^N} M_{j_1 j_2} M_{j_2 j_3} \dots M_{j_{N-1} j_N} (T_{j_1} + \dots T_{j_N})^{-1} \mathcal{Q}_{j_N j_1} dx' d\xi'$$

where we plug $\tau = 0$.

This latter equals to

$$(4.14) \quad \Omega(\tau) = (2\pi h)^{1-d} \iint \text{tr} \Upsilon(L(x', \xi', h^{-1}\tau)) \mathcal{Q}(x', \xi, \tau) dx' d\xi'$$

where L is defined by $e^{iL} = S$ and Υ is defined by (6.2.63).

Theorem 4.3. *Consider two Schrödinger operators A_1 and A_2 with all periodic trajectories on levels close to τ , satisfying (3.1). Further, let (3.5) and (4.11) be fulfilled. Then asymptotics*

$$(4.15) \quad \text{tr } \Gamma(e(., ., \tau)) = \varkappa_0(\tau)h^{-d} + (\varkappa_1(\tau) + \Omega(\tau))h^{1-d} + o(h^{1-d})$$

holds.

Example 4.4. Consider example 3.6 in the new conditions to Hamiltonian flows. We can use results of that example immediately to treat billiards with complete internal reflections (thus non-branching).

To treat branching billiards we need to calculate eigenvalues of the matrix (\varkappa_{jk}) . It follows from calculations of example 3.6 that $\varkappa_{22} = -\varkappa_{11}$; thus we arrive to

$$(4.16) \quad \lambda = e^{\pm i\varphi}, \quad \varphi = \arccos \varkappa_{11} = \arccos((\omega - 1)(\omega + 1)^{-1});$$

then obviously (4.11) is fulfilled and asymptotics (4.15) holds.

Remark 4.5. On the contrary assume that (4.11) is not fulfilled. Then there will be eigenvalues of high multiplicity or clusters of eigenvalues located in $o(h)$ -vicinities of solutions τ of equation $\det(S - 1) = 0$. One can rewrite this equation as

$$(4.17) \quad \cos(-\varphi + \frac{1}{2}\tau h^{-1}(T_1 + T_2)) = \cos \alpha \cdot \cos(-\psi + \frac{1}{2}\tau h^{-1}(T_1 - T_2))$$

as $M = \begin{pmatrix} e^{i(\phi+\psi)} \cos \alpha & e^{i(\phi+\chi)} \sin \alpha \\ -e^{i(\phi-\chi)} \sin \alpha & e^{i(\phi-\psi)} \cos \alpha \end{pmatrix}$ which is the general form of the unitary matrix.

The following problem seems to be a difficult one and worth of publication:

Problem 4.6. Prove the same results as flows $\Psi_{j,t}$ are assumed to be periodic only on one energy level τ rather than on neighboring levels.

On the contrary, the following problem seems to be relatively easy:

Problem 4.7. (i) Prove the same results as there are more than two manifolds X_j $j = 1, \dots, m'$ with the all billiards periodic.

(ii) Extend these results to the case when there are also manifolds X_j $j = m' + 1, \dots, m$ with almost all billiards non-periodic.

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